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(Supplement to BRL MR No. 1477)

DYNAMICS OF LIQUID-FILLED SHELL: AIDS FOR DESIGNERS

by

John T. Frasier

December 1967

U. S. ARMY MATERIEL COMMAND
BALLISTIC RESEARCH LABORATORIES
ABERDEEN PROVING GROUND, MARYLAND

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RDT&E Project No. 1T014501A33D

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ABSTRACT

Ballistic Research Laboratories Memorandum Report 1477 (Karpov, May 1963) includes a discussion of Stewartson's stability analysis of liquid-filled projectiles, and "Tables of Poles and Residues" needed for quantitative design use of the analysis. Various design problems show that an extended tabulation is needed, and the present report provides this extension. The new tabulation (APPENDIX B) covers a non-dimensional frequency range of 0.06 to 0.70 in increments of 0.02 and a range of cavity fill ratios of 20 to 100 percent in maximum increments of ten percent.

A brief description of Stewartson's analysis is also given in this report. Emphasis is placed on the physical significance of the assumptions and results of the theory rather than its mathematical detail. The intent is to provide the novice designer of liquid-filled shell with an appreciation and first working knowledge of the analysis. Additionally, the summary of the theory is used to point out the significant advances that have been made in understanding liquid-filled shell problems since the publication of Ballistic Research Laboratories Memorandum Report No. 1477.

TABLE OF CONTENTS

I. INTRODUCTION

In Ballistics Research Laboratories Memorandum Report 1477, May 1963, Karpov presented recommendations and aids for designers of liquid-filled shell. He advised that Stewartson's[1]* analysis concerning the flight stability of spinning projectiles carrying liquid in cylindrical cavities represented the best, *a priori* means for design against flight instability of liquid-filled shell. The theory demonstrates that unstable flights are the result of resonance between oscillations in the liquid and the nutational motion of the shell. To aid designers in use of the theory, Karpov summarized the development of Stewartson's instability criterion, and provided an extensive numerical tabulation of the "poles and residues" required for quantitative design work. This tabulation, in conjunction with Stewartson's criterion, allows designers to calculate the adverse combinations of physical and geometrical properties of a shell and its liquid filler that will render an unstable flight. Two parameters govern the range of conditions covered by the tables of poles and residues (1) the liquid fill-ratio of the cavity, and (2) certain non-dimensional eigenfrequencies (natural frequencies) of the liquid. In Karpov's tabulation[2], fill-ratios from 50 to 100 percent are encompassed, while the frequency range is 0.00 to 0.50.

Superscript numbers denote references which may be found on page 31.

A number of instances have occurred where the coverage of the tables given by Karpov was insufficient for quantitative design. Hence, calculations were undertaken to extend the tables to include fill-ratios of 20 to 100 percent (in steps of 5 and 10 percent) and frequencies from 0.00 to 0.70 (in steps of 0.02). The primary purpose of the present report is to provide designers with the extended tabulation. As a convenience to users of the tables a brief summary of Stewartson's theory is also included. The assumptions, capabilities, limitations, and important formulae of the theory are covered, but not in extensive mathematical detail. By so doing, we hope to provide a useful reference for the designer familiar with the stability problems of liquid-filled shell, and to give the novice designer an appreciation and first working knowledge of the basic theory.

Above we mentioned Karpov's advice to make use of Stewartson's theory to design liquid-filled shell. There is no reason to alter this recommendation except to state it more emphatically and to take cognizance of recent advances in our understanding of the factors influencing the behavior of these shell. The theory remains the most effective basis for design of well behaved liquid-filled projectiles. Its elements should be understood by designers and its results should be put to use at every opportunity. Furthermore, advantage should be taken of the increased capability for design analysis achieved through the research efforts of Karpov[2], Wedemeyer[3], and Scott[4]. Stewartson's basic theory deals only

with shell containing inviscid, fully-spinning liquids in cylindrical cavities. However, the latter investigators have shown how to treat problems involving:

(1) Viscous effects in the liquid filler[2b, c, 3c].

(2) Certain types of non-cylindrical cavities, including those with profiles similar to the ogival shape of the conventional artillery shell,[2f, 3d]

(3) Liquid spin-up effects[2e] (by a semi empirical approach).

It is important that designers know of these advances and we intend this report to have the secondary purpose of providing an awareness of them. This will be done in the course of our summary of Stewartson's theory by discussion (but not development of formulae) and reference to important publications. An extensive theoretical account of the advances is not appropriate here. Such detail would likely defeat our purpose of supplying a working appreciation of Stewartson's basic theory. Furthermore, a Design Handbook for Liquid-Filled Projectiles is to be published and will cover in detail all phases of liquid-filled projectile theory and design.* Accordingly, this report is prepared as an interim supplement to BRL Memorandum Report 1477.

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*The handbook, "Liquid-Filled Projectile Design," is to be published in the Army Materiel Command Engineering Handbook series.

II. STEWARTSON'S THEORY OF STABILITY

Stewartson's theory concerns the flight stability of a spinning shell with a right circular cylindrical cavity either wholly or partially filled with liquid. Results of the theory show that growth of the nutational component of the projectile's yaw is possible under adverse combinations of the geometrical and physical characteristics of the projectile and its liquid filler. The mechanism producing this instability is resonance between the nutational frequency of the shell and certain of the natural frequencies of the liquid. When resonance occurs, oscillations of the liquid produce a periodic moment (couple) on the shell casing and lead to the growth in yaw.

The theory is very valuable for several reasons. First, it provides a clear understanding of the physical phenomena through which liquids produce instability of spinning projectiles; namely, the resonance behavior mentioned above. This basic mechanism applies to cavities of all geometries and not just cylinders. With this knowledge, we are better prepared even for *ad hoc* design approaches. Second, the theory provides a quantitative means for designing against instability in liquid-filled shell provided the various requirements placed on the shell allow use of a cylindrical cavity*. Of course, Stewartson's theory is not universal in application but is based upon assumptions and stipulations that define the situations for which it is valid. Here we will review the most

*Actually, we can now design for some non-cylindrical cavities.

important of these factors and attempt to point out their physical significance. Our primary attention is to be directed to the initial theory. However, when appropriate, significant research advances by other investigators will be mentioned and references to their work cited.

We begin our summary of the theory by stating and discussing the conditions and assumptions on which it is founded. Certain of the stipulations are absolute requirements in that if they are not satisfied the theory is invalid. Others are taken as a matter of convenience to simplify the analysis and to clarify the role of the liquid in causing flight instability. An example of an assumption of the latter type is that the overturning moment is the only significant aerodynamic force or moment acting on the shell. Drag, etc., can be included in the analysis but are not essential to its development. Therefore, we shall neglect them to maintain focus on the basic features of interaction between the shell and its liquid. Distinctions between the two types of assumptions will become clear in the course of discussion.

A. <u>Assumptions of Stewartson's Analysis.</u>

(1) <u>The cavity in the shell is a right circular cylinder whose axis is parallel to the spin axis of the projectile.</u> Immediately we see that the theory is restricted to shell with cylindrical cavities. Hence, direct quantitative application of its results is limited to a single geometrical shape. Wedemeyer[1d] however, has achieved a modification

of Stewartson's theory through which it is possible to design for other shapes; specifically, cavities whose radii change slowly along their length. As a consequence, we are able to perform analyses on many practical cavity geometries. To use Wedemeyer's modification, one must have a working knowledge of the basic Stewartson analysis.

(2) <u>The aerodynamic overturning moment is the only external force or moment affecting the flight of the shell</u>. We explained above that this assumption is taken for convenience. Gravity and other aerodynamic effects can be taken into account, but are not essential to the theory. By considering only the aerodynamic overturning moment, we will be able to design against instability due to the liquid.

(3) <u>The shell flies with constant translational velocity and spin</u>. This assumption is consistent with (2) above but also implies that the liquid does not influence the spin rate of the projectile. In practice these conditions are never satisfied, of course. For a period after the projectile leaves the gun muzzle, the shell's spin decreases because it must spin-up the liquid. Subsequent to liquid spin-up, the shell

experiences spin decrease due to drag. Furthermore, the projectile encounters translational drag. In general however, once the transient phase of liquid spin-up is complete, the drag effects are sufficiently small for the current assumption to be reasonable for most projectiles.

Finally, we remark that this assumption deals with the shell casing~ not with the liquid~and relates to the equation of motion written for the projectile. Assumptions (4) and (5) below concern the motion of the liquid.

(4) The gross motion of the liquid is a rigid body translation and spin identical to the translation and spin of the projectile. This assumption, in conjunction with (3), restricts our considerations to the full spin condition of the liquid. The theory does not consider situations where the shell casing and liquid have unequal rigid body spins, nor does it take account of variations in the spin of the liquid.

Wedemeyer[21] and Scott[22] have performed analyses that allow us to calculate the time required for the liquid to spin-up after a projectile leaves the muzzle. Hence, in practical situations, we can determine when Stewartson's full-spin assumption becomes valid. Usually, it is soon after exit from the muzzle. Furthermore, a semi-empirical analysis is available to determine whether instability is likely during the transient spin-up process[23]. The possibility of this latter phenomena should be guarded against in the design of all liquid-filled shell.

(5) The spin of the liquid and the dimensions of the cylindrical cavity satisfy the condition

$$a^2\Omega^2 \gg qc \qquad (1)$$

where

a = cavity radius, inches.

Ω = spin rate of the projectile (and therefore the liquid), rad/sec.

q = magnitude of the resolved gravity and drag vectors, in/sec^2.

$2c$ = height of the cavity, inches.

The physical significance of this assumption is that centrifugal forces exerted on the liquid due to its spin far outshadow any forces imposed by gravity or drag. A consequence of the assumption is that the liquid (except when the cavity is completely filled) has the shape of a cylinder with a hollow, cylindrical core*.

Equation (1) must be satisfied for Stewartson's theory to be used. If a shell experiences high drag along with a low spin rate there is a possibility the relation will be violated. Then the liquid will not have a cylindrical core, but will develop a paraboloidal surface. The designer should always verify Equation (1) is satisfied to avoid imprudent application of the theory.

*Actually a paraboloid whose vertex is far from the shell.

It should be understood that the current assumption is distinct from assumption (2). The former condition concerns external forces and moments acting on the shell casing and their effect on the motion of the shell. The present assumption concerns the effect of gravity and drag on the behavior of the liquid.

(6) The mass of the liquid is small compared to the total mass of the shell. This assumption is one of convenience. It is satisfied for many shell and simplifies the equations of motion for the liquid-shell system. We use it here for these reasons.

(7) The liquid is incompressible and inviscid. The assumption of incompressibility is reasonable for the liquids encountered in actual projectiles. Viscous effects, however, can influence the behavior of liquid-filled projectiles. Fortunately, Wedemeyer[3c] has provided an analysis to account for these effects. His analysis involves a boundary layer correction to the basic, inviscid theory of Stewartson. Here we consider only the fundamental inviscid case and recommend that designers become familiar with Wedemeyer's work once they have Stewartson's theory well in hand.

(8) The final assumptions concern the nature of any variations to the rigid body translation and spin of the shell and liquid. Any disturbance to the shell's motion is restricted to small amplitude

perturbations superposed on its gross translation and spin. Correspondingly, the liquid is assumed to experience only small amplitude perturbations to its large scale translation and spin. The assumption about the shell is the familiar small yaw situation associated with the linearized equations of yawing motion. Similarly, the assumption imposed on the liquid linearizes the equations describing its behavior. By virtue of linearization, the equations for the liquid motions can be solved and their result incorporated into the equations of the motion for the shell.

B. Results of Stewartson's Analysis.

All the basic assumptions and conditions underlying Stewartson's theory were presented above. From these assumptions, we can make a qualitative statement of our problem. Namely, determine the conditions for which a symmetric, rapidly spinning projectile will experience flight instability as a consequence of having liquid (at full spin) in a cylindrical cavity along its axis. Stewartson attacked this problem in two phases and it seems most effective to describe his analysis in a similar fashion. First, he considered the behavior of the liquid in a state of rapid rotation within a container that could perform motions similar to those of the yawing motion of a shell. He then combined the problem solution he found for the liquid with the equations of motion for a shell. Upon analysis of the resulting equations it was found that under certain adverse conditions the yaw of the shell will grow without limit.

To describe the behavior of the liquid, we remember that it is confined in a container and that its basic motion involves rigid body spin about an axis with fixed direction. Upon assuming that the axis of the container is subjected to a small disturbance similar to the yawing motion of a shell, it is necessary that the liquid also experiences a disturbance to its basic motion because it must follow the walls of the cavity. Stewartson's solution shows that the liquid conforms to the cavity motion through the exitation of small amplitude oscillations superposed on the rigid body motion. There is an infinite number of discrete frequencies for these oscillations - the natural frequencies (or eigenfrequencies) of the spinning liquid. For an arbitrary motion of the container all the natural frequencies will be excited, but in varying degrees. If, however, the container performs a yawing motion at certain of the eigenfrequencies of the liquid, oscillations at this frequency become predominant, that is, a condition of resonance is established. As we shall describe later, it is this resonance that leads to instability of a liquid filled projectile.

We should emphasize that the oscillations performed by the liquid are of small amplitude. Sloshing does not occur, but a wave pattern is established in the longitudinal, radial, and circumferential directions of the cavity and there are mode numbers* associated with each direction. For problems of projectile stability, an infinity of the possible

*These can be thought of as fundamental wave patterns and harmonics.

longitudinal and radial modes are significant theoretically, but only the first circumferential mode is important. This circumstance is a result of the fact that the pressure fluctuations produced in the liquid by this mode lead to a periodic couple (with the same frequency as that of the oscillating liquid) on the walls of the container. It is this couple which renders a projectile unstable.

Now, we must explain how the natural frequencies of the liquid are determined. Stewartson's theory gives these frequencies in a complicated relation of the form

$$\tau_{nj} = \frac{\omega_{nj}}{\Omega} = f_n \left[\frac{c}{a(2j+1)}, \frac{b^2}{a^2}\right]; \quad n = 1, 2, 3, \cdots \quad j = 0, 1, 2, \cdots \tag{2}$$

where

n = radial mode number (the number of nodes in the radial wave pattern),

j = longitudinal wave number (2j+1 = number of nodes in the longitudinal wave pattern),

ω_{nj} = natural frequency of the njth mode,

τ_{nj} = the non-dimensional eigenfrequency of the njth mode,

2a = diameter of the cavity,

2b = diameter of the cylindrical air core,

2c = cavity length.

Several aspects of Equation (1) should be noted. First, the eigenfrequencies of the liquid are dependent upon the cavity geometry through the ratios c/a and b^2/a^2. The ratio b^2/a^2 is the air volume in the cavity expressed as a fraction of the total cavity volume. Hence $(1 - b^2/a^2)$ is the fraction of the cavity occupied by liquid. Next, we note that the eigenfrequencies depend upon the longitudinal mode number through the ratio $c/a(2j+1)$ appearing as a variable in f_n. This is a fortunate circumstance, because once τ_{nj} is known for a set of fixed values of $c/a(2j+1)$, b^2/a^2, and n, the eigenfrequencies are known for all longitudinal modes for which $c/a(2j+1)$ equals the set value. Finally, Equation (2) shows the frequencies are linearly related to Ω, that is, τ_{nj} is independent of Ω.

As mentioned above, the function f_n in Equation (1) is a complicated one and it must be evaluated numerically through machine determination of the poles (singularities) of another equation appearing in the Stewartson analysis. The eigenfrequency (poles) portion of Stewartson's Tables is no more than a numerical tabulation of selected values of τ_{nj}, $c/a(2j+1)$, b^2/a^2, and n satisfying Equation (1).

Appendix B contains the tabulation of f_n. Arrangement of the Tables is as follows. Each sheet is for a fixed value of the parameter b^2/a^2 and contains a column headed τ, and column pairs headed $c/a(2j+1)$ and $2R$* for several values of n. To find the value of τ_{nj} satisfying Equation (1)

*The significance of this column will be explained later.

for given values of b^2/a^2, $c/a(2j+1)$, and n we proceed as follows. Turn to the sheet for the specified value of b^2/a^2 and pick out the column pair headed by the proper value of n. Next, read down the $c/a(2j+1)$ column to the assigned value and read to the left to determine τ_{sj}. When the values of b^2/a^2 or $\frac{c}{a(2j+1)}$ do not appear explicitly in the Table, linear interpolation is to be used.

This completes our discussion of the natural frequencies of the spinning liquid. We have shown how these frequencies depend upon the cavity geometry, i.e., (c/a and b/a) and demonstrated the method for finding the frequencies from Stewartson's Tables. Now, we turn to the questions of how and when instability of a liquid-filled shell is produced by the oscillating liquid. To begin we recall our assumptions that the overturning moment is the only significant aerodynamic force or moment acting on the shell and that we are dealing with small yaws. Under these conditions, the motion of the shell (without the liquid) is governed by the relations

$$\lambda = \lambda_o e^{i\Omega\tau t} \tag{3}$$

$$I_y\tau^2 - I_x\tau + \frac{I_x^2}{4I_y s} = 0 \tag{4}$$

where I_x and I_y are, respectively, the axial and transverse moments of inertia of the shell, and

λ is the complex yaw,

t is time,

τ is the non-dimensional frequency of the motion of the shell,

s is the gyroscopic stability factor.

Equation (3) represents the form of the motion of the shell and the values of τ are provided by solution of Equation (4). Since Equation (4) is quadratic, the latter are found easily:

$$\tau_n = \frac{1}{2}\frac{I_x}{I_y}(1 + \sigma) \qquad \text{Nutational frequency} \qquad (5)$$

$$\tau_p = \frac{1}{2}\frac{I_x}{I_y}(1 - \sigma) \qquad \text{Precessional frequency} \qquad (6)$$

where

$$\sigma = \sqrt{1 - \frac{1}{s}} \ ,$$

$$s = \frac{\bar{\nu}^2}{4M} \ ,$$

$$\bar{\nu} = \frac{I_x}{I_y}\frac{2\pi}{n} \ ,$$

$$M = \frac{\rho_a S d}{2m} K_y^{-2} C_{M_\alpha} \ ,$$

ρ_a = air density ,

S = reference area of shell, usually the cross-sectional area ,

n = twist of rifling, calibers per turn,

d = diameter of the shell,

m = mass of the shell,

$K_a^{-2} = md^2/I_y$,

C_{M_α} = aerodynamic overturning moment coefficient.

It is advantageous to recall here that the shell is stable (the yaw does not grow with time) so long as τ_1 and τ_2 are real quantities. To achieve this situation we must have s greater than one, a familiar condition for the gyroscopic stability of a projectile. If s is less than one, σ and therefore τ_1 and τ_2 become imaginary, and an exponential growth of yaw occurs.

Earlier, we stated that the oscillating liquid produces a moment on the casing of the projectile. Now we must describe that moment in functional form and modify Equation (4) to include its effect. Stewartson showed that when the frequency, τ, of the projectile is near any one of the natural frequencies, τ_{sj}, of the liquid, the moment applied to the shell casing is given by

$$\frac{M_L}{\Omega^2} = - \frac{\rho a^5 [2R(\tau_{sj})]^2/4c}{\tau - \tau_{sj}} , \qquad (7)$$

where M_l is the moment exerted on the shell by the liquid,

τ is the frequency of motion of the shell,

ρ is the density of the liquid,

$R(\tau_{nj})$ is a (small) constant depending upon τ_{nj} and is always positive. The quantity R_{nj} is the "residue" of the Tables of poles and residues, and we see from Equation (7) that it governs the magnitude of the liquid-moment for a given cavity and frequency, τ_{nj}. Every possible frequency and modal configuration of the liquid involves a specific value of R_{nj} and these values are listed in the tables adjacent to $c/a(2j+1)$. When τ_{nj} is obtained from the tables, the corresponding value of R_{nj} is obtained by reading to the right of the value for $\frac{c}{a(2j+1)}$. With regard to the residue, it is convenient to point out a significant feature of its behavior as seen in the tables. Namely, for any specific value of frequency, τ_{nj}, the residue decreases greatly for each successively larger value of n (i.e., R_{nj} decreases with increasing radial mode number). Thus, the higher radial modes produce relatively weak liquid-moments. In practice it has been found that modes beyond n = 2 are seldom strong enough to cause shell to be unstable.

The moment due to the liquid is a forcing function on the motion of the shell. Thus to account for the presence of the liquid in the equation of motion of the shell we add Equation (7) to the right-hand-side of Equation (4) and obtain

$$I_y\tau^2 - I_x\tau + \frac{I_x^2}{4I_y s} = -\frac{\rho a^6[2R(\tau_0)]^2/4c}{\tau - \tau_0}, \tag{8}$$

where for convenience, τ_0 has been written in place of τ_s, to emphasize that we are now thinking of a specific fluid frequency. By solving Equation (8) for τ, the frequency of the shell's motion and the conditions under which the liquid can produce an unstable flight are determined, (that is, the conditions for which τ has an imaginary part assuming, of course, that $s > 1$). Here, we shall only summarize the results of this solution. It is found that when τ_c is close to τ_p, the precessional frequency, no instability occurs. However, if τ_0 is near the nutational frequency, τ_n, Equation (8) has the roots

$$\tau = \left(\frac{\tau_n + \tau_0}{2}\right) \pm \sqrt{\left(\frac{\tau_n - \tau_0}{2}\right)^2 - \frac{\rho a^6[2R(\tau_0)]^2}{4cI_x\sigma}} \tag{9}$$

The condition for <u>instability</u> is provided immediately by Equation (9). When the quantity under the radical is negative, τ has a negative imaginary part as we see by substituting (9) into (3):

$$\lambda = \lambda_0 \exp\left\{i\Omega\left(\frac{\tau_n + \tau_0}{2}\right)\tau - i\Omega\sqrt{\left(\frac{\tau_n - \tau_0}{2}\right)^2 - \frac{\rho a^6(2R)^2}{4cI_x\sigma}}\, t\right\} \tag{10}$$

$$= \lambda_0 \exp\left[i\Omega\left(\frac{\tau_n + \tau_0}{2}\right) t\right] \exp\left(\alpha\Omega t\right),$$

where

$$\alpha = \sqrt{\frac{\rho a^6(2R)^2}{4cI_x\sigma} - \left(\frac{\tau_n - \tau_0}{2}\right)^2}$$

Thus, an exponential growth of the nutational component of yaw occurs when

$$\left(\frac{\tau_s - \tau_0}{2}\right)^2 - \frac{\rho a^6 (2R)^2}{4 c I_x \sigma} < 0 \; .$$

or, written more conveniently, when

$$-1 < (\tau_0 - \tau_s)/S^{\frac{1}{2}} < 1 \qquad \text{The condition for instability} \qquad (11)$$

where S, "Stewartson's Parameter," is

$$S = \frac{\rho a^5 (2R)^2}{I_x \sigma (c/a)} \; .$$

When Equation (11) is satisfied, the rate of growth of yaw is [Equation (10)]

$$\alpha = \tfrac{1}{2} \sqrt{S - (\tau_0 - \tau_s)^2} \; , \qquad (12)$$

or

$$\frac{2}{\sqrt{S}}\,\alpha = \sqrt{1 - \left(\frac{\tau_0 - \tau_s}{\sqrt{S}}\right)^2}$$

Figure 1 is a plot of $(2/S^{\frac{1}{2}})\alpha$ against $(\tau_0 - \tau_s)/S^{\frac{1}{2}}$. This resonance curve, as well as examination of Equations (11) and (12), shows the

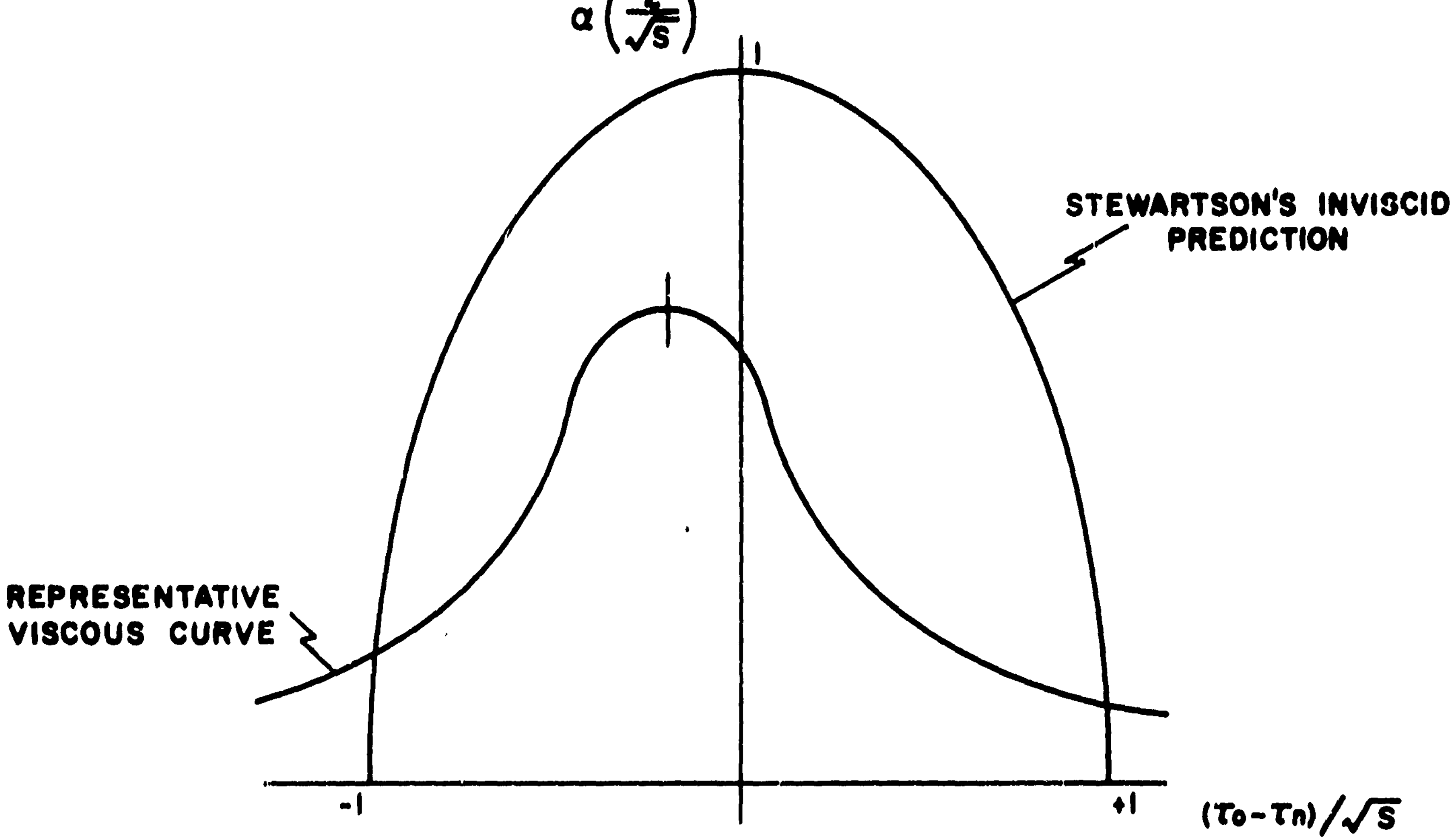

Figure 1. Resonance curves for liquid-filled projectiles

instability to be strongest when $\tau_0 = \tau_s$. For non-zero values of $\tau_0 - \tau_s$, the yaw growth rate decreases until it vanishes for $|\tau_0 - \tau_s| = S^{\frac{1}{2}}$.

Equations (11) and (12) are the basic results of the Stewartson theory. They permit us to calculate the conditions producing instability in a given projectile (and therefore the means to avoid these instabilities) and to calculate the strength of the instability.

Appendix A provides examples of the use of the theory in shell design. In regard to use of the theory for quantitative design, a cautionary note is advisable here. It concerns the theoretical limits of $\pm$ 1 appearing in Equation (11) which were derived under the assumption that the liquid is inviscid. Wedemeyer [3c] and Karpov [2b] have shown that viscous effects can exert a significant influence on the shape of the resonance curve. The influence is demonstrated qualitatively in Figure 1 where a representative resonance curve for a viscous liquid is drawn along with the curve provided by the inviscid analysis. Viscosity is seen to decrease the peak value of α, however, it also broadens the curve. In fact, with viscosity, the curve never becomes zero or negative and the liquid always tends to cause the yaw to grow. However, on the wings of the curve, α is very small and can be overcome by aerodynamic damping in actual practice. Hence, the limits (in Equation (11)) to be used in practice must be established by the shape of the viscous resonance curve and the magnitude of aerodynamic damping. We mention the effects of viscosity here because indiscriminate use of the theoretical limits of $\pm$ 1 appearing in Equation (11) could lead to an

improperly designed shell. Every designer is advised to study Wedemeyer's viscous correction to Stewartson's theory once he has become familiar with the basic, inviscid analysis.

The previous discussion has surveyed Stewartson's analysis of the instability of liquid-filled shell with cylindrical cavities. The mechanism producing instability is resonance between the nutational frequency of the projectile and certain of the eigenfrequencies of the spinning liquid. For these eigenfrequencies, the pressure fluctuations in the liquid produce a periodic moment on the walls of the cavity and this moment continuously increases the yaw of the projectile. The tables of poles and residues calculated from the analysis provide the means to determine the liquid cylinder dimensions that will cause a given shell to be unstable, and to determine the strength of the instability. Thus, the theory enables us to make *a priori* design against instabilities in liquid-filled shell when a cylindrical cavity is employed. If other design considerations allow the use of such a cavity, it should be used to assure a well-flying projectile. When requirements demand other than a cylindrical cavity, every attempt should be made to have the container satisfy the conditions* used in Wedemeyer's analysis for non-cylindrical cavities. Then the designer still can perform a quantitative study of his shell.

There are situations where it is necessary to use cavity geometries not covered by the analyses of Stewartson and Wedemeyer.

*That the cavity radius changes slowly as a function of cavity length.

An *ad hoc* design approach is then necessary. However, even in these cases Stewartson's analysis can be put to qualitative use. For even though the numerical results of the theory do not apply, the mechanism for instability remains the same: resonance between natural frequencies of the shell and liquid. With this understanding, the astute designer possesses valuable guidance for selecting and modifying cavity geometries to achieve a stable projectile.

III. CONCLUSION

Stewartson's theory and the work of Karpov, Wedemeyer, and Scott provide the shell designer with powerful tools for design against flight instability of liquid-filled projectiles. To use these tools, it is necessary to have a firm understanding of the physical basis and significance of the theoretical results, and the numerical tables required for quantitative work. In this report, we hope to have supplied a part of both. To give the tables is a simple task, and those contained in Appendix B should satisfy the requirements of virtually all design problems where the theory is applicable. To give an adequate description of the theory is a much more difficult problem. The full mathematical analysis is both complex and tedious, and we recognize that its detailed pursuit is beyond the time demands placed on most designers. In contrast to the mathematics, however, the physical phenomena associated with instability of liquid-filled shell are relatively simple, and familiar to the designer. Thus,

we have attempted to summarize Stewartson's theory by stressing the physical significance of its basic assumptions and results. Once the designer has these factors well in hand, he is prepared to apply Stewartson's results to problems of shell design, and to study the more recent advances achieved through liquid-filled shell research.

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APPENDIX A

Examples of Design of Liquid-Filled Shell

The design problems contained in this appendix have been extracted without change from BRL Memo Report 1477. They provide lucid examples of the design procedure for shell with cylindrical cavities. When working through the examples, the reader will note that Stewartson's criterion for instability (Equation (11)) is used with the limits -3.9, and +2.7. These were recommended for design practice prior to Wedemeyer's analysis of viscous effects. Here we have not changed the limits to emphasize that the theoretical (inviscid) ones of $\pm$ 1 often require modification as discussed in the body of this report.

To illustrate the use of the tables, we may consider two cases: a) new shell, and b) existing shell. For the new shell it is important to know, a priori, the cavity dimensions which will give rise to resonance between the fluid oscillations and the nutational frequency of the shell. The designer, therefore, should avoid these dimensions. If the existing shell is being adapted for carrying liquids, it is important to inquire whether its cavity is such as to lead to resonance and, presumably, bad flight.

Consider the case of a new shell. The designer should estimate the moments of inertia of the empty shell, I_x and I_y, and the gyroscopic stability factor, s, from which to compute σ. With these he computes the nutational frequency

$$\tau_s = \frac{1}{2} \frac{I_x}{I_y} (1 + \sigma)$$

It might be adequate, at this stage, to use approximate formulae for I_x and I_y.

$$I_x = 0.14\ md^2$$

$$I_y = 0.5I_x + 0.0594\ mL^2$$

where m = mass of the shell

d = diameter

L = overall length

The gyroscopic stability factor

$$s = \frac{\bar{\nu}^2}{4M}$$

where $$\bar{\nu} = \frac{I_x}{I_y} \frac{2\pi}{n},$$

and n = twist of rifling in calibers per turn. M can be written

$$M = \frac{\rho_a S d}{2m} K_a^{-2} \; (cp - cg) \; C_{N_\alpha}$$

where cp = center of pressure, in calibers

cg = center of gravity, in calibers

C_{N_α} = normal force coefficient

cp and C_{N_α} depend only on the exterior shape of the shell and, hence, remain invariant with the changes in the cavity dimensions.

EXAMPLE 1:

Let us consider, as an example, the 105mm chemical shell for which we have either estimated or measured the following characteristics:

105mm Chemical Shell

I_x = 0.56 lbs ft^2

I_y = 5.56 lbs ft^2

s = 1.2; σ = .41

$\therefore \tau_s$ = 0.07

The geometry of the cavity is right circular cylinder with the diameter $a = 0.13$ ft, and with the fineness ratio, c/a, so selected as to avoid resonance condition. The cavity is to be filled to 95% by a liquid of density $\rho = 62.4\ \text{lbs/ft}^3$.

The problem is to find which fineness ratios will lead to resonance and, hence, are to be avoided.

To do this, we turn to the table for $b^2/a^2 = .05$ (95% full cavity) and locate on the line $\tau_o = \tau_s = 0.07$ the corresponding values of $\frac{c/a}{2j+1}$ and associated residue, 2R, in each of the three column pairs. We find the following:

For $\tau = .07$	$\frac{c/a}{2j+1}$	2R
1st column pair	1.070^5	.212
2nd column pair	$.509^5$	.0252
3rd column pair	.320	.0067

Therefore, the resonance will occur at the following c/a values:

$$(c/a)_1 = 1.070^5 \quad (2j+1)$$
$$(c/a)_2 = .509^5 \quad (2j+1)$$
$$(c/a)_3 = .320 \quad (2j+1)$$

j	$(c/a)_1$	$(c/a)_2$	$(c/a)_3$
0	1.08	(.51)	(.32)
1	3.24	1.53	.96
2	5.39	2.55	1.60
3		3.57	2.24
4		4.59	2.88
5			3.52
6			4.16

It is to be noted that, for simplicity, we have assumed that the value of the nutational frequency, τ_a, remained constant while we changed c/a. In practice, of course, τ_a will change because I_x, I_y and σ all will change, albeit slowly, with changes in c/a. But these can be taken into account at more detailed examination of the situation in the vicinity of the desired c/a ratio. The present rough survey stakes out only the danger areas.

The above table shows a fairly large number of fineness ratios which are to be avoided in order to escape resonance. However, the situation is not as bad as it looks. The third column, $(c/a)_3$, contains a greater number of entries. But because of the very small residues associated with this column, the coincidence of the actual value of c/a with tabulated values must be very precise for resonance to occur. This can be shown as follows. Using the residues given above, we compute the quantity

$\frac{1}{2R}\left(\frac{I_x\sigma}{\rho a^5}\right)^{\frac{1}{2}}$ for each column pair. Equation (11) can be rewritten as*:

$$\text{1st} \quad -3.9 < 50\sqrt{\frac{c}{a}}\left[\left(\frac{c}{a}\right)_1 - \left(\frac{c}{a}\right)\right] < 2.7$$

$$\text{2nd} \quad -3.9 < 400\sqrt{\frac{c}{a}}\left[\left(\frac{c}{a}\right)_2 - \left(\frac{c}{a}\right)\right] < 2.7$$

$$\text{3rd} \quad -3.9 < 1500\sqrt{\frac{c}{a}}\left[\left(\frac{c}{a}\right)_3 - \left(\frac{c}{a}\right)\right] < 2.7$$

Where $\left(\frac{c}{a}\right)_i$ are the tabulated values as given above, for each column pair, and $\left(\frac{c}{a}\right)$ the actual designed value. It is clear that in order to satisfy the above instability conditions the difference in the vicinity of tabulated fineness ratios and the actual designed values must be very small for the 3rd column, less so for the 2nd column, and still less critical for the 1st column. In practice, therefore, the coincidence with tabulated values in the 3rd column is likely to be purely fortuitous because one usually cannot design the cavity with the required precision. The fineness ratios appearing in the 1st column are the most important and should, therefore, be avoided.

<u>EXAMPLE 2:</u>

For the second case, we may consider the same 105mm chemical shell.

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*Note the limits of -3.9 and 2.7 used here rather than the theoretical limits of $\pm$ 1. See comment at beginning of this appendix.

Let us suppose that the fineness ratio of its cavity is $\frac{c}{a} = 3.2$. We have found already that this fineness ratio should be avoided. But, as an illustrative example, we shall proceed with the analysis of this case.

$$\text{Let } A = \frac{I_1 c}{\rho a^5} \cdot \frac{c}{a} = \frac{(.56)(.41)}{(62.4)(.37 \times 10^{-4}} \quad 3.2 = 317.1$$

$$\therefore \sqrt{A} = 17.8$$

The condition for instability is, therefore,

$$-3.9 < \frac{\tau_0 - .07}{2R} (17.8) < 2.7$$

$$\text{Let } B \equiv \frac{\tau_0 - \tau_c}{2R} \sqrt{A}$$

Since the cavity is to be filled to 95% we use the table for $b^2/a^2 = .05$.

Compute: From Table $b^2/a^2 = .05$

j	$\frac{c/a}{2j+1}$	τ_0	$2R$	B
0	3.2	-	-	-
1	1.067	.064	.194	-.55
2	0.640	.23	.104	26.4
3	0.457	.305	.0457	92
4	0.356	.346	.0162	83
5	0.291	-	.0182	-
6	0.246	-	.0036	-

The results show that for this cavity, c/a = 3.2, and 95% fill the shell is predicted to be unstable for j = 1.

The hydrodynamic moment is proportional to the residues at the poles. Hence, only the leading poles, lower j values and associated larger residues, are the most important.

One of the possible remedies is to try to change the geometry of the cavity or, more specifically, the fineness ratio c/a in the vicinity of 3.2. The following table illustrates the effect of such a change. For this illustration again, the inertial properties of the shell were kept constant and only c/a changed.

The Effect of Changing c/a on Stability. 105mm Shell

		Values of "B"			
j	c/a = 2.8	3.0	3.2	3.4	3.6
1	22	-.42	-.55	2.3	3.2
2	21	26	26	28	29
3	39	90	92	72	5
4	-73	51	83	97	101
5				153	37

Thus, this shell is unstable for c/a = 3.2, as previously shown, but also is dangerously close to instability for values of c/a up to 3.6. For c/a = 3.57 for example, it is unstable for j = 3 for which the "B" value is -.76.

Another possible remedy is to alter the air space or fill ratio. The following table conveys some sense of sensitivity of instability condition to various air spaces.

<u>The Effect of Changing Air Space on Stability.</u> 105mm Shell

$$c/a = 3.2, \quad j = 1, \quad \frac{c/a}{2j+1} = 1.067$$

b^2/a^2	"B"
.00	-.87
.02	-.87
.05	-.55
.10	.16
.15	1.04
.20	1.8

With the fineness ratio of the cavity of 3.2, the shell is unstable for fill ratios from 80% to 100%. For this shell, therefore, with a fineness ratio of the cavity of 3.2, changing the loading conditions is not an effective means of remedying a bad situation.

Another example:

20mm shell		90% full; $b^2/a^2 = 0.10$	
I_x	42.4 gms-cm²	τ_a	= 0.155
I_y	251.6 gms-cm²	c/a	2.68
σ	0.84	a	0.77
ρ	3 gms/cm³		

$$\therefore A = \frac{42.4(.84)}{3(.271)}\ 2.7 = 118.3$$

and instability criteria becomes:

$$-3.9 < \frac{\tau_o - \tau_2}{2R}\, 10.9 < 2.7$$

j	$\frac{c/a}{2j+1}$	τ_c	$2R$	B
0	2.68	-	-	-
1	.893	.44	.2683	11.6
2	.536	.245	.0523	-2.11
3	.383	.26	.0305	38
4	.298	.08	.0065	-126
5	.244	-	-	-

The shell, therefore, is stable if one uses Stewartson's limits of $\pm$ 1 but is unstable for $j = 2$, if one uses the limits -3.9 and + 2.7.

APPENDIX B

EXTENDED TABLES OF POLES AND RESIDUES

The tables for b^2/a^2 of 0.00, 0.01 and 0.03 were not extended, and are reproduced from BRL Memorandum Report 1477.

τ_c	n = 1: C/(2J+1)A	n = 1: 2R	n = 2: C/(2J+1)A	n = 2: 2R	n = 3: C/(2J+1)A	n = 3: 2R
.00	.995	.000	.478	.0000	.310	.0000
.02	1.018	.058	.490	.0070	.319	.00[illegible]9
.04	1.042	.118	.503	.0144	.327	.00[illegible]0
.06	1.066	.181	.516	.0223	.336	.0062
.08	1.091	.246	.530	.0307	.345	.0086
.10	1.117	.313	.544	.0396	.355	.0111
.12	1.144	.382	.559	.0491	.364	.0139
.14	1.[illegible]2	.454	.574	.0591	.375	.0168
.16	1.201	.528	.590	.0697	.385	.0198
.18	1.231	.604	.607	.0809	.397	.023[illegible]
.20	1.262	.682	.624	.0928	.408	.0266
.22	1.294	.762	.642	.1054	.420	.030[illegible]
.24	1.328	.845	.661	.1187	.433	.0344
.26	1.363	.930	.680	.1328	.446	.0387
.28	1.399	1.017	.700	.1478	.460	.0433
.30	1.437	1.107	.722	.1636	.475	.0481
.32	1.478	1.200	.745	.1804	.490	.0533
.34	1.521	1.295	.769	.1981	.506	.0589
.36	1.565	1.392	.7[illegible]4	.2169	.523	.0649
.38	1.612	1.491	.820	.2369	.541	.07[illegible]4
.40	1.662	1.593	.848	.2581	.561	.0783
.42	1.715	1.698	.878	.2805	.582	.0853
.44	1.771	1.805	.910	.3043	.603	.09[illegible]
.46	1.831	1.914	.944	.3296	.626	.102[illegible]
.48	1.895	2.026	.980	.3566	.651	.1118
.50	1.963	2.142	1.019	.3853	.678	.1220

$b^2/a^2 = 0.00$

	$n = 1$		$n = 2$		$n = 3$	
r_0	$\frac{c}{(2j+1)a}$	2R	$\frac{c}{(2j+1)a}$	2R	$\frac{c}{(2j+1)a}$	2R
0.00	0.995	0.000	0.477	0.0000	0.309	0.0000
0.02	1.018	0.058	0.490	0.0070	0.317	0.0019
0.04	1.042	0.118	0.503	0.0144	0.325	0.0040
0.06	1.066	0.181	0.516	0.0222	0.334	0.0062
0.08	1.091	0.246	0.530	0.0305	0.343	0.0086
0.10	1.117	0.313	0.544	0.0393	0.353	0.0111
0.12	1.144	0.382	0.559	0.0486	0.363	0.0138
0.14	1.172	0.453	0.574	0.0584	0.373	0.0167
0.16	1.201	0.527	0.590	0.0688	0.384	0.0198
0.18	1.232	0.603	0.607	0.0798	0.395	0.0231
0.20	1.263	0.681	0.625	0.0915	0.407	0.0266
0.22	1.296	0.761	0.643	0.1038	0.420	0.0304
0.24	1.330	0.844	0.662	0.1168	0.433	0.0345
0.26	1.366	0.929	0.682	0.1305	0.447	0.0388
0.28	1.404	1.016	0.704	0.1450	0.461	0.0434
0.30	1.443	1.105	0.727	0.1603	0.477	0.0484
0.32	1.485	1.195	0.751	0.1765	0.493	0.0538
0.34	1.529	1.289	0.776	0.1935	0.510	0.0595
0.36	1.576	1.384	0.803	0.2114	0.529	0.0658
0.38	1.626	1.481	0.832	0.2301	0.549	0.0724
0.40	1.680	1.578	0.864	0.2496	0.571	0.0795
0.42	1.737	1.676	0.898	0.2699	0.595	0.0874
0.44	1.799	1.772	0.935	0.2905	0.621	0.0958
0.46	1.868	1.864	0.978	0.3110	0.650	0.1050
0.48	1.945	1.945	1.025	0.3305	0.685	0.1150
0.50	2.034	2.005	1.085	0.3475	0.721	0.1262

$b^2/\varepsilon^2 = 0.01$

	n = 1		n = 2		n = 3	
τ_o	C/(2J+1)A	2R	C/(2J+1)A	2R	C/(2J+1)A	2R
.00	.994	.000	[illegible]	.0000	.305	.0000
.02	[illegible]	[illegible]	[illegible]	.0060	.314	.0020
.04	[illegible]	[illegible]	[illegible]	[illegible]	.322	.0040
.06	1.066	[illegible]	[illegible]	.0219	.331	.0062
.08	1.091	.245	.528	.0302	.340	.0085
.10	1.117	.312	.542	.0389	.349	.0110
	[illegible]	.381	.556	.0481	.359	.0136
[illegible]	[illegible]	[illegible]	.572	.0578	.369	.0165
.16	[illegible]	.526	.588	.0680	.380	.0195
.18	1.232	.602	.605	.0788	.391	.0228
.20	[illegible]	.680	.623	.0902	.403	.0264
.22	1.297	.760	.642	.1023	.416	.0302
.24	1.332	.849	.662	[illegible]	.429	.0343
.26	1.369	[illegible]	.682	.1285	.443	.0386
.28	1.407	[illegible]	.704	.1428	.458	.0433
.30	1.448	[illegible]	.727	.1579	.473	.0483
.32	1.491	1.194	.752	.1738	.490	.0538
.34	1.537	1.286	.778	.1906	.507	.0596
.36	1.586	1.380	.806	.2083	.526	.0660
.38	1.638	1.475	.837	.2268	.547	.0730
.40	1.695	1.570	.871	.2462	.569	.0807
.42	1.757	1.664	.909	.2665	[illegible]	.0892
.44	1.825	1.755	.951	.2876	.620	.0985
.46	1.902	1.839	.998	.3095	.650	.1090
.48	1.990	1.909	1.051	.3321	.683	.1207
.50	2.097	1.953	1.112	.3553	.719	.1338
.52	2.234	1.951	[illegible]	.3811	.762	.1500
.54	2.459	1.877	1.295	.4160	.812	.1700
.56	2.841	1.759	[illegible]	.4820	.869	.1961
.58			1.647	.5419	.936	.2311
.60			1.893	.9860	1.012	.2774
.62			[illegible]	1.4607	1.094	.3348
.64			2.309	1.9151	1.181	[illegible]
.66			2.483	2.2951	1.272	[illegible]
.68			2.667	2.6144	1.368	.5400
.70			2.855	2.8913	1.471	.6116

$b^2/a^2 = 0.02$

τ_c	n = 1		n = 2		n = 3	
	$\frac{c}{(2j+1)a}$	2R	$\frac{c}{(2j+1)a}$	2R	$\frac{c}{(2j+1)a}$	2R
0.00	0.993	0.000	0.472	0.0000	0.301	0.0000
0.02	1.016	0.058	0.485	0.0069	0.309	0.0019
0.04	1.040	0.118	0.498	0.0141	0.317	0.0039
0.06	1.065	0.180	0.511	0.0218	0.326	0.0060
0.08	1.090	0.245	0.525	0.0299	0.335	0.0083
0.10	1.116	0.312	0.539	0.0384	0.344	0.0107
0.12	1.144	0.381	0.554	0.0474	0.354	0.0134
0.14	1.172	0.453	0.569	0.0570	0.364	0.0162
0.16	1.201	0.526	0.585	0.0671	0.375	0.0192
0.18	1.232	0.602	0.602	0.0777	0.386	0.0224
0.20	1.264	0.680	0.620	0.0890	0.398	0.0259
0.22	1.298	0.760	0.639	0.1009	0.410	0.0297
0.24	1.334	0.842	0.659	0.1135	0.423	0.0337
0.26	1.371	0.927	0.680	0.1268	0.437	0.0380
0.28	1.410	1.013	0.702	0.1409	0.452	0.0427
0.30	1.452	1.102	0.726	0.1558	0.467	0.0477
0.32	1.495	1.193	0.751	0.1716	0.483	0.0532
0.34	1.543	1.285	0.778	0.1882	0.501	0.0591
0.36	1.593	1.378	0.807	0.2058	0.520	0.0655
0.38	1.648	1.473	0.838	0.2245	0.540	0.0725
0.40	1.708	1.568	0.872	0.2444	0.562	0.0805
0.42	1.774	1.661	0.910	0.2654	0.586	0.0888
0.44	1.847	1.751	0.952	0.2877	0.612	0.0985
0.46	1.930	1.834	1.000	0.3115	0.640	0.1089
0.48	2.029	1.905	1.055	0.3374	0.672	0.1209
0.50	2.150	1.956	1.120	0.3654	0.707	0.1348

$b^2/a^2 = 0.03$

τ_o	n = 1: C/(2J+1)A	n = 1: 2R	n = 2: C/(2J+1)A	n = 2: 2R	n = 3: C/(2J+1)A	n = 3: 2R
.00	.991	.000	.465	.0000	.291	.0000
.02	1.014	.058	.477	.0067	.299	.0018
.04	1.038	.118	.490	.0138	.307	.0037
.06	1.062	.180	.503	.0212	.316	.0057
.08	1.088	.245	.516	.0292	.324	.0078
.10	1.114	.311	.530	.0373	.333	.0101
.12	1.141	.381	.545	.0460	.343	.0126
.14	1.169	.452	.560	.0552	.353	.0153
.16	1.199	.525	.576	.0650	.363	.0182
.18	1.231	.601	.593	.0753	.374	.0213
.20	1.264	.679	.611	.0863	.385	.0246
.22	1.299	.759	.630	.0979	.397	.0282
.24	1.335	.841	.650	.1101	.410	.0321
.26	1.373	.926	.671	.1231	.423	.0362
.28	1.413	1.013	.693	.1368	.437	.0407
.30	1.456	1.102	.717	.1514	.452	.0455
.32	1.502	1.193	.742	.1668	.468	.0508
.34	1.551	1.286	.769	.1833	.485	.0565
.36	1.605	1.381	.798	.2009	.503	.0627
.38	1.663	1.477	.830	.2199	.522	.0696
.40	1.727	1.574	.865	.2405	.543	.0771
.42	1.799	1.670	.903	.2625	.565	.0855
.44	1.880	1.765	.945	.2867	.591	.0948
.46	1.974	1.857	.994	.3134	.617	.1051
.48	2.087	1.944	1.048	.3453	.647	.1167
.50	2.229	2.026	1.109	.3778	.679	.1296
.52	2.421	2.105	1.180	.4198	.715	.1447
.54	2.711	2.213	1.271	.4742	.755	.1622
.56	3.255	2.493	1.379	.5503	.800	.1828
.58	4.949	4.128	1.510	.6627	.851	.2072
.60			1.665	.8314	.907	.2363
.62			1.837	1.0676	.971	.2712
.64			2.016	1.3576	1.042	.3122
.66			2.198	1.6702	1.121	.3596
.68			2.382	1.9800	1.209	.4151
.70			2.575	2.2757	1.305	.4726

$r^2/a^2 = 0.05$

T_o	n = 1		n = 2		n = 3	
	C/(2J+1)A	2R	C/(2J+1)A	2R	C/(2J+1)A	2R
.00	[illegible]	.000	[illegible]	.0000	.267	.0000
.02	1.003	.057	[illegible]	.0061	.274	.0015
.04	1.027	.117	.455	.0125	.282	.0031
.06	1.051	.179	.477	.0193	.289	.0047
.08	1.072	.243	.493	.0265	.297	.0065
.10	1.103	.309	.505	.0339	.306	.0085
.12	1.130	.377	.517	.0417	.314	.0106
.14	1.158	.447	.532	.0500	.323	.0128
.16	1.189	.521	.547	.0588	.333	.0153
.18	1.221	.596	.563	.0682	.343	.0179
.20	1.255	.675	.580	.0781	.353	.0207
.22	1.290	.755	.598	.0885	.364	.0238
.24	1.328	.839	.617	.0998	.375	.0270
.26	1.368	.925	.637	.1116	.387	.0306
.28	1.410	1.014	.659	.1242	.400	.0344
.30	1.454	1.105	.681	.1376	.414	.0385
.32	1.502	1.199	.705	.1520	.428	.0430
.34	1.555	1.296	.731	.1675	.443	.0479
.36	1.612	1.397	.759	.1843	.459	.0531
.38	1.676	1.502	.789	.2025	.476	.0589
.40	1.749	1.611	.821	.2225	.495	.0653
.42	1.829	1.725	[illegible]	[illegible]	[illegible]	.0723
.44	1.921	1.841	[illegible]	.2683	.537	.0803
.46	2.029	1.968	[illegible]	.2950	.560	.0885
.48	2.160	2.107	[illegible]	.3253	.585	.0979
.50	2.327	2.268	1.041	.3602	.612	.1084
.52	2.549	2.472	[illegible]	.4011	.643	.1203
.54	2.876	2.779	[illegible]	.4502	.675	.1337
.56	3.434	3.384	1.255	.5105	.712	.1489
.58	4.773	5.366	1.345	.5869	.753	[illegible]
.60			1.454	.6860	.798	.1855
.62			1.573	.8133	.849	.2097
.64			1.718	.9755	.906	.2368
.66			1.875	1.1707	.971	.2682
.68			2.041	1.3947	1.044	.3047
.70			2.222	1.6378	1.126	.3469

[illegible]$^2/a^2 = 0.30$

τ_o	n = 1 C/(2J+1)A	n = 1 2R	n = 2 C/(2J+1)A	n = 2 2R	n = 3 C/(2J+1)A	n = 3 2R
.00	.966	.000	.415	.0000	.245	.0000
.02	.988	.057	.425	.0054	.251	.0012
.04	1.011	.115	.436	.0111	.258	.0025
.06	1.035	.176	[illegible]	.0170	.265	.0039
.08	1.060	.239	.460	.0233	.272	.0054
.10	1.086	.304	.473	.0299	.280	.0070
.12	1.113	.371	.486	.0368	.288	.0087
.14	1.141	.441	.500	.0441	.296	.0106
.16	1.171	.514	.514	.0519	.305	.0126
.18	1.203	.590	.529	.0601	.314	.0148
.20	1.237	.668	.545	.0689	.323	.0171
.22	1.273	.750	.562	.0782	.333	.0196
.24	1.311	.834	.580	.0880	.343	.0223
.26	1.351	.921	.598	.0985	.354	.0253
.28	1.394	1.011	.618	.1097	.366	.0284
.30	1.439	1.104	.639	.1217	.378	.0318
.32	1.488	1.201	.662	.1346	.391	.0355
.34	1.542	1.303	.685	.1484	.405	.0395
.36	1.601	1.410	.711	.1635	.419	.0439
.38	1.668	1.523	.739	.1799	.435	.0487
.40	1.743	1.645	.769	.1978	.451	.0539
.42	1.826	1.773	.801	.2173	.469	.0596
.44	1.922	1.911	.836	.2389	.488	.0658
.46	2.035	2.064	.875	.2630	.509	.0727
.48	2.172	2.241	.918	.2900	.531	.0803
.50	2.349	2.458	.964	.3205	.556	.0887
.52	2.569	2.739	1.017	.3556	.582	.0981
.54	2.888	3.149	1.075	.3954	.611	.1087
.56	3.400	3.885	1.142	.4447	.643	.1206
.58	4.448	5.768	1.218	.5029	.678	.1340
.60	9.421	21.765	1.305	.5735	.717	.1493
.62			1.404	.6607	.760	.1658
.64			1.517	.7689	.809	.1870
.66			1.645	.9006	.864	.2103
.68			1.790	1.0582	.927	.2373
.70			1.951	1.2408	.998	.2688

$b^2/a^2 = 0.15$

T_0	n = 1: C/(2J+1)A	n = 1: 2R	n = 2: C/(2J+1)A	n = 2: 2R	n = 3: C/(2J+1)A	n = 3: 2R
.00	[illegible]	.000	[illegible]	.0000	.224	.0000
.02	[illegible]	.056	[illegible]	.0048	.230	.0010
.04	[illegible]	.113	.408	.0097	.236	.0020
.06	1.015	.172	.419	.0149	.242	.0032
.08	1.039	.234	.450	.0202	.249	.0044
.10	1.065	.298	[illegible]	[illegible]	.256	.0057
.12	1.091	.364	[illegible]	[illegible]	.263	.0071
.14	[illegible]	.433	[illegible]	.0382	.271	.0086
.16	1.149	.504	[illegible]	.0450	.279	.0103
.18	1.180	.579	[illegible]	.0521	.287	.0121
.20	1.214	.657	.503	.0597	[illegible]	.0140
.22	1.249	.739	[illegible]	.0677	.305	.0160
.24	1.287	.823	.541	.0762	.314	.0183
.26	1.327	.910	.558	.0853	.324	.0207
.28	1.369	1.001	.577	.0951	.334	.0232
.30	1.415	1.096	.596	.1055	.345	.0260
.32	1.464	1.195	.617	.1167	.357	.0291
.34	1.518	1.299	.639	.1288	[illegible]	.0323
.36	1.577	1.411	.662	.1419	[illegible]	.0359
.38	1.644	1.530	.688	.1561	.391	.0398
.40	1.719	1.660	[illegible]	.1717	.412	.0440
.42	1.803	1.800	[illegible]	.1888	.428	.0487
.44	1.899	1.954	[illegible]	.2077	.445	.0537
.46	2.011	2.129	[illegible]	.2285	.464	.0593
.48	2.147	2.331	[illegible]	.2516	.484	.0654
.50	2.315	2.576	[illegible]	.2774	.505	.0722
.52	2.532	2.897	.937	.3070	.529	.0797
.54	2.851	3.358	[illegible]	.3409	.555	[illegible]
.56	3.287	4.128	1.045	.3799	.583	[illegible]
.58	4.114	5.731	[illegible]	.4257	.614	[illegible]
.60	6.509	12.660	[illegible]	.4801	.648	[illegible]
.62			1.255	.5454	.687	[illegible]
.64			1.360	.6239	.729	.1495
.66			1.469	.7193	.778	.1674
.68			1.592	.8342	.832	.1885
.70			1.733	.9706	.894	.2125

$b^2/a^2 = 0.20$

τ_o	n = 1: C/(2J+1)A	n = 1: 2R	n = 2: C/(2J+1)A	n = 2: 2R	n = 3: C/(2J+1)A	n = 3: 2R
.00	.925	.000	.360	.0000	.205	.0000
.02	.946	.054	.370	.0041	.210	.0008
.04	.968	.110	.379	.0083	.216	.0016
.06	.991	.168	.389	.0127	.222	.0026
.08	1.015	.229	.400	.0173	.228	.0035
.10	1.039	.290	.411	.0221	.234	.0046
.12	1.065	.354	.422	.0272	.241	.0058
.14	1.092	.422	.434	.0326	.248	.0070
.16	1.122	.492	.446	.0383	.255	.0083
.18	1.153	.565	.459	.0444	.262	.0098
.20	1.185	.642	.473	.0509	.270	.0113
.22	1.220	.722	.488	.0577	.278	.0130
.24	1.257	.806	.503	.0650	.287	.0148
.26	1.296	.893	.519	.0728	.296	.0168
.28	1.338	.983	.536	.0811	.306	.0189
.30	1.383	1.078	.553	.0900	.316	.0211
.32	1.432	1.178	.572	.0996	.326	.0236
.34	1.485	1.285	.593	.1100	.338	.0263
.36	1.543	1.398	.614	.1211	.350	.0291
.38	1.609	1.522	.637	.1333	.362	.0323
.40	1.683	1.657	.662	.1466	.376	.0357
.42	1.765	1.803	.689	.1612	.390	.0395
.44	1.859	1.966	.718	.1772	.406	.0436
.46	1.970	2.150	.749	.1949	.423	.0481
.48	2.100	2.366	.783	.2144	.441	.0530
.50	2.261	2.630	.820	.2361	.460	.0584
.52	2.465	2.970	.861	.2607	.481	.0645
.54	2.739	3.445	.907	.2887	.504	.0712
.56	3.138	4.190	.957	.3206	.530	.0787
.58	3.802	5.616	1.013	.3573	.557	.0871
.60	5.307	9.816	1.076	.4004	.588	.0966
.62			1.148	.4511	.622	.1074
.64			1.225	.5111	.660	.1197
.66			1.322	.5834	.703	.1339
.68			1.428	.6695	.751	.1502
.70			1.550	.7727	.806	.1692

$b^2/a^2 = 0.25$

τ_o	n = 1: C/(2J+1)A	n = 1: 2R	n = 2: C/(2J+1)A	n = 2: 2R	n = 3: C/(2J+1)A	n = 3: 2R
.00	[illegible]	.000	.333	.0000	.187	.0000
.02	.920	.052	.342	.0034	.192	.0006
.04	.941	.106	.351	.0070	.197	.0013
.06	.963	.162	.360	.0106	.202	.0020
.08	.987	.220	.370	.0145	.208	.0028
.10	1.011	.281	.380	.0186	.213	.0037
.12	1.036	.343	.391	.0229	.219	.0046
.14	1.063	.408	.402	.0274	.225	.0056
.16	1.091	.476	.413	.0322	.232	.0067
.18	1.121	.548	.425	.0373	.239	.0079
.20	1.153	.623	.438	.0428	.246	.0091
.22	1.187	.701	.451	.0485	.254	.0105
.24	1.223	.783	.465	.0547	.262	.0119
.26	1.261	.869	.480	.0612	.270	.0135
.28	1.302	.958	.495	.0682	.279	.0152
.30	1.346	1.052	.512	.0757	.288	.0170
.32	1.393	1.152	.529	.0838	.297	.0190
.34	1.445	1.258	.548	.0925	.308	.0211
.36	1.502	1.372	.567	.1019	.318	.0235
.38	1.565	1.496	.588	.1122	.330	.0260
.40	1.637	1.633	.611	.1234	.342	.0288
.42	1.717	1.781	.635	.1356	.356	.0318
.44	1.808	1.948	.661	.1490	.370	.0351
.46	1.913	2.138	.690	.1638	.385	.0387
.48	2.037	2.360	.720	.1801	.401	.0426
.50	2.188	2.629	.754	.1932	.419	.0470
.52	2.378	2.972	[illegible]	.2185	.438	.0518
.54	2.625	[illegible]	[illegible]	.2415	.458	.0571
.56	2.973	4.134	.875	.2676	.481	.0631
.58	3.517	5.361	.925	.2974	.506	.0698
.60	4.564	8.244	[illegible]	.3319	.533	.0774
.62	8.380	24.961	1.042	.3718	.564	.0859
.64			1.113	.4189	.598	.0957
.66			1.193	.4746	.636	.1068
.68			1.286	.5410	.679	.1197
.70			1.391	.6197	.728	.1346

$b^2/a^2 = 0.30$

τ_o	n = 1 C/(2J+1)A	n = 1 2R	n = 2 C/(2J+1)A	n = 2 2R	n = 3 C/(2J+1)A	n = 3 2R
.00	.872	.000	.307	.0000	.170	.0000
.02	.892	.050	.315	.0028	.174	.0005
.04	.912	.102	.323	.0057	.179	.0010
.06	.934	.155	.332	.0088	.184	.0016
.08	.957	.211	.341	.0120	.189	.0023
.10	.980	.269	.350	.0154	.194	.0029
.12	1.004	.329	.360	.0189	.200	.0037
.14	1.030	.392	.370	.02[illegible]	.205	.0045
.16	1.057	.458	.380	.0267	.211	.0053
.18	1.086	.527	.391	.0309	.217	.0063
.20	1.117	.600	.403	.0354	.224	.0073
.22	1.150	.676	.415	.0402	.231	.0083
.24	1.184	.755	.428	.0453	.238	.0095
.26	1.221	.838	.441	.0507	.245	.0108
.28	1.261	.926	.456	.0565	.253	.0121
.30	1.303	1.018	.471	.0628	.261	.0136
.32	1.349	1.116	.487	.0695	.270	.0151
.34	1.399	1.220	.503	.0767	.280	.0169
.36	1.454	1.333	.521	.0845	.289	.0187
.38	1.515	1.457	.541	.0930	.300	.0207
.40	1.584	1.593	.561	.1023	.311	.0229
.42	1.659	1.741	.583	.1124	.323	.0253
.44	1.7[illegible]	1.908	.607	.1235	.336	.0279
.46	1.846	2.097	.632	.1357	.349	.0308
.48	1.963	2.317	.660	.1492	.364	.0339
.50	2.104	2.582	.690	.1640	.380	.0374
.52	2.277	2.916	.723	.1807	.397	.0412
.54	2.500	3.359	.759	.1995	.416	.0455
.56	2.802	3.996	.79[illegible]	.2207	.436	.0502
.58	3.250	5.039	.84[illegible]	.2448	.458	.0555
.60	4.022	7.143	.892	.2724	.483	.06[illegible]5
.62	5.920	14.119	.945	.3043	.510	.0682
.64			1.003	.3415	.541	.0759
.66			1.078	.3851	.575	.0847
.68			1.159	.4366	.613	.09[illegible]
.70			1.251	.4980	.657	.1065

$\beta^2/\epsilon^2 = 0.35$

τ_o	n = 1		n = 2		n = 3	
	C/(2J+1)A	2R	C/(2J+1)A	2R	C/(2J+1)A	2R
.00	.842	.000	.281	.0000	.154	.0000
.02	.861	.047	.288	.0023	.158	.0004
.04	.881	.097	.296	.0047	.162	.0008
.06	.901	.148	.304	.0072	.166	.0013
.08	.923	.201	.312	.0098	.171	.0018
.10	.945	.256	.320	.0125	.176	.0023
.12	.969	.313	.329	.0154	.181	.0029
.14	.994	.373	.338	.0185	.186	.0035
.16	1.020	.435	.348	.0218	.191	.0042
.18	1.048	.503	.358	.0252	.197	.0049
.20	1.077	.572	.369	.0289	.203	.0057
.22	1.109	.645	.380	.0328	.209	.0066
.24	1.142	.722	.392	.0369	.215	.0075
.26	1.178	.802	.404	.0414	.222	.0085
.28	1.215	.886	.417	.0461	.229	.0095
.30	1.256	.975	.431	.0512	.237	.0107
.32	1.300	1.070	.445	.0567	.245	.0119
.34	1.348	1.172	.460	.0626	.253	.0133
.36	1.401	1.282	.477	.0690	.262	.0147
.38	1.459	1.402	.494	.0759	.271	.0163
.40	1.524	1.535	.513	.0835	.281	.0181
.42	1.595	1.680	.533	.0917	.292	.0199
.44	1.677	1.842	.554	.1008	.304	.0220
.46	1.771	2.026	.577	.1107	.316	.0243
.48	1.881	2.240	.602	.1217	.329	.0267
.50	2.010	2.497	.629	.1337	.343	.0294
.52	2.163	2.814	.658	.1473	.359	.0324
.54	2.367	3.226	.691	.1624	.376	.0358
.56	2.629	3.799	.726	.1795	.394	.0395
.58	3.000	4.675	.763	.1989	.414	.0436
.60	3.591	6.270	.805	.2210	.436	.0483
.62	4.764	10.204	.857	.2463	.460	.0536
.64	9.819	39.504	.911	.2757	.488	.0596
.66			.973	.3100	.518	.0664
.68			1.043	.3502	.552	.0744
.70			1.124	.3980	.592	.0835

$b^2/a^2 = 0.40$

T_o	n = 1: C/(2J+1)A	n = 1: 2R	n = 2: C/(2J+1)A	n = 2: 2R	n = 3: C/(2J+1)A	n = 3: 2R
.00	.810	.000	.255	.0000	.138	.0000
.02	.828	.045	.262	.0018	.142	.0003
.04	.847	.092	.269	.0037	.146	.0006
.06	.866	.140	.276	.0057	.150	.0010
.08	.887	.190	.284	.0078	.154	.0014
.10	.908	.242	.291	.0100	.158	.0018
.12	.931	.296	.299	.0123	.163	.0022
.14	.954	.353	.308	.0148	.167	.0027
.16	.979	.412	.317	.0174	.172	.0032
.18	1.006	.475	.326	.0202	.177	.0038
.20	1.034	.541	.335	.0231	.182	.0044
.22	1.064	.610	.345	.0263	.188	.0051
.24	1.096	.683	.356	.0296	.194	.0058
.26	1.130	.759	.367	.0332	.200	.0066
.28	1.166	.840	.379	.0370	.206	.0074
.30	1.205	.925	.391	.0410	.213	.0083
.32	1.247	1.016	.405	.0454	.220	.0092
.34	1.292	1.114	.418	.0502	.228	.0103
.36	1.342	1.220	.433	.0553	.236	.0114
.38	1.397	1.335	.449	.0609	.244	.0127
.40	1.459	1.462	.466	.0669	.253	.0140
.42	1.528	1.604	.484	.0735	.263	.0155
.44	1.603	1.760	.503	.0808	.273	.0171
.46	1.691	1.937	.524	.0888	.284	.0188
.48	1.792	2.141	.546	.0975	.296	.0207
.50	1.910	2.380	.570	.1072	.309	.0228
.52	2.052	2.676	.596	.1180	.323	.0252
.54	2.229	3.052	.625	.1301	.338	.0277
.56	2.456	3.560	.657	.1437	.354	.0306
.58	2.765	4.299	.691	.1590	.372	.0338
.60	3.227	5.527	.730	.1765	.392	.0374
.62	4.028	8.043	.773	.1965	.413	.0415
.64	6.051	16.751	.821	.2196	.438	.0461
.66			.875	.2464	.465	.0514
.68			.937	.2778	.495	.0576
.70			1.007	.3148	.530	.0646

$c^2/a^2 = 0.45$

τ_o	n = 1		n = 2		n = 3	
	C/(2J+1)A	2R	C/(2J+1)A	2R	C/(2J+1)A	2R
.00	.775	.000	.230	.0000	.123	.0000
.02	.792	.042	.236	.0014	.127	.0002
.04	.810	.085	.242	.0029	.130	.0005
.06	.828	.131	.249	.0045	.134	.0007
.08	.848	.178	.256	.0061	.137	.0010
.10	.869	.226	.263	.0078	.141	.0014
.12	.890	.277	.270	.0097	.145	.0017
.14	.912	.330	.278	.0116	.149	.0021
.16	.936	.386	.286	.0137	.154	.0025
.18	.961	.444	.294	.0158	.158	.0029
.20	.988	.506	.303	.0181	.163	.0034
.22	1.017	.571	.312	.0206	.168	.0039
.24	1.047	.640	.321	.0232	.173	.0044
.26	1.079	.712	.331	.0250	.179	.0050
.28	1.113	.788	.342	.0290	.184	.0056
.30	1.150	.869	.353	.0322	.190	.0063
.32	1.190	.955	.365	.0357	.197	.0070
.34	1.233	1.047	.377	.0394	.203	.0078
.36	1.280	1.147	.391	.0434	.211	.0087
.38	1.331	1.256	.405	.0478	.218	.0096
.40	1.389	1.377	.420	.0526	.226	.0107
.42	1.453	1.510	.436	.0578	.235	.0118
.44	1.525	1.659	.453	.0635	.244	.0130
.46	1.606	1.828	.472	.0697	.254	.0143
.48	1.698	2.018	.492	.0766	.264	.0158
.50	1.803	2.236	.513	.0842	.276	.0174
.52	1.931	2.507	.537	.0926	.288	.0191
.54	2.088	2.847	.562	.1021	.301	[illegible]
.56	2.284	3.290	.590	.1127	.316	.0233
.58	2.543	3.910	.621	.1247	.332	.0257
.60	2.909	4.866	.655	.1383	.349	.0285
.62	3.487	6.588	.693	.1539	.369	.0316
.64	4.625	10.812	.735	.1718	.390	.0351
.66	9.260	39.918	.783	.1926	.414	.0391
.68			.836	.2167	.442	.0438
.70			.898	.2453	.472	.0492

$b^2/a^2 = 0.50$

τ_o	n = 1		n = 2		n = 3	
	C/(2J+1)A	2R	C/(2J+1)A	2R	C/(2J+1)A	2R
.00	.697	.000	.182	.0000	.096	.0000
.02	.712	.035	[illegible]	.0008	.098	.0001
.04	.728	.071	.191	.0017	.101	.0003
.06	.744	.109	.196	.0025	.104	.0004
.08	.762	.149	.202	.0035	.107	.0006
.10	.780	.190	.207	.0044	.110	.0007
.12	.799	.233	.213	.0055	.113	.0009
.14	.819	.278	.219	.0066	.116	.0011
.16	.840	.325	.225	.0077	.119	.0013
.18	.862	.375	.232	.0090	.123	.0016
.20	.886	.427	.239	.0103	.126	.0018
.22	.911	.482	.246	.0117	.130	.0021
.24	.937	.541	.254	.0132	.134	.0024
.26	.966	.602	.261	.0147	.138	.0027
.28	.996	.667	.270	.0164	.143	.0030
.30	1.028	.736	.278	.0183	.148	.0034
.32	1.063	.810	.288	.0202	.152	.0038
.34	1.100	.889	.298	.0224	.158	.0042
.36	1.141	.975	.308	.0247	.163	.0047
.38	1.185	1.068	.319	.0272	.169	.0052
.40	1.234	1.170	.331	.0299	.175	.0057
.42	1.288	1.283	.343	.0328	.182	.0063
.44	1.348	1.409	.357	.0361	.189	.0070
.46	1.416	1.549	.371	.0397	.197	.0077
.48	1.491	1.708	.387	.0436	.205	.0085
.50	1.577	1.886	.403	.0479	.214	.0093
.52	1.678	2.102	.422	.0527	.223	.0103
.54	1.798	2.364	.441	.0581	.233	.0113
.56	1.942	2.689	.463	.0642	.245	.0125
.58	2.124	3.114	.487	.0710	.257	.0138
.60	2.360	3.702	.513	.0787	.270	.0153
.62	2.687	4.585	.541	.0875	.285	.0170
.64	3.191	6.133	.574	.0976	.302	.0189
.66	4.116	9.591	.610	.1092	.320	.0210
.68	6.948	25.388	.650	.1228	.341	.0235
.70			.697	.1388	.365	.0264

[illegible]$^2/\varepsilon^2 = 0.60$

	n = 1		n = 2		n = 3	
τ_o	C/(2J+1)A	2R	C/(2J+1)A	2R	C/(2J+1)A	2R
.00	.606	.000	.134	.0000	.070	.0000
.02	.619	.028	.138	.0004	.072	.0001
.04	.633	.057	.142	.0008	.074	.0001
.06	.647	.086	.145	.0012	.076	.0002
.08	.661	.117	.149	.0017	.078	.0003
.10	.677	.149	.154	.0021	.080	.0003
.12	.693	.183	.158	.0026	.082	.0004
.14	.710	.218	.162	.0032	.084	.0005
.16	.728	.256	.167	.0037	.087	.0006
.18	.747	.295	.172	.0043	.090	.0007
.20	.767	.336	.177	.0050	.092	.0008
.22	.783	.380	.182	.0056	.095	.0009
.24	.811	.426	.188	.0063	.098	.0011
.26	.835	.474	.194	.0071	.101	.0012
.28	.860	.526	.200	.0079	.104	.0014
.30	.887	.580	.206	.0088	.108	.0016
.32	.916	.639	.213	.0098	.111	.0017
.34	948	.702	.220	.0108	.115	.0019
.36	.981	.769	.228	.0119	.119	.0021
.38	1.018	.843	.236	.0131	.123	.0024
.40	1.058	.923	.245	.0145	.128	.0026
.42	1.102	1.012	.254	.0159	.133	.0029
.44	1.151	1.109	.264	.0175	.138	.0032
.46	1.204	1.217	.275	.0192	.143	.0035
.48	1.264	1.338	.286	.0211	.149	.0039
.50	1.330	1.473	.298	.0232	.156	.0043
.52	1.407	1.632	.312	.0256	.163	.0047
.54	1.496	1.818	.326	.0282	.170	.0052
.56	1.600	2.042	.342	.0312	.178	.0058
.58	1.725	2.319	.359	.0345	.187	.0064
.60	1.878	2.674	.378	.0382	.197	.0070
.62	2.074	3.152	.399	.0425	.208	.0078
.64	2.337	3.845	.422	.0474	.220	.0087
.66	2.718	4.960	.448	.0531	.233	.0097
.68	3.347	7.116	.478	.0597	.248	.0108
.70	4.729	13.311	.511	.0675	.265	.0122

$b^2/a^2 = 0.70$

τ_o	n = 1: C/(2J+1)A	n = 1: 2R	n = 2: C/(2J+1)A	n = 2: 2R
.00	.496	.000	.088	.0000
.02	.507	.019	.091	.0001
.04	.517	.039	.093	.0003
.06	.529	.060	.096	.0004
.08	.541	.081	.098	.0006
.10	.553	.104	.101	.0008
.12	.566	.127	.104	.0009
.14	.580	.152	.107	.0011
.16	.594	.178	.110	.0013
.18	.609	.205	.113	.0016
.20	.625	.234	.117	.0018
.22	.642	.264	.120	.0020
.24	.660	.296	.124	.0023
.26	.679	.330	.128	.0026
.28	.699	.366	.132	.0029
.30	.720	.405	.136	.0032
.32	.743	.445	.141	.0035
.34	.768	.489	.145	.0039
.36	.794	.536	.150	.0043
.38	.822	.587	.156	.0048
.40	.853	.643	.162	.0052
.42	.387	.703	.168	.0058
.44	.923	.770	.174	.0063
.46	.963	.843	.181	.0070
.48	1.007	.924	.189	.0077
.50	1.056	1.013	.197	.0084
.52	1.110	1.116	.205	.0093
.54	1.172	1.234	.215	.0102
.56	1.243	1.371	.225	.0113
.58	1.325	1.534	.237	.0125
.60	1.422	1.731	.249	.0139
.62	1.537	1.975	.263	.0154
.64	1.681	2.294	.278	.0172
.66	1.863	2.721	.295	.0193
.68	2.108	3.346	.314	.0217
.70	2.466	4.367	.335	.0246

$b^2/a^2 = 0.80$

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13. ABSTRACT

Ballistic Research Laboratories Memorandum Report 1477 (Karpov, May 1963) includes a discussion of Stewartson's stability analysis of liquid-filled projectiles, and "Tables of Poles and Residues" needed for quantitative design use of the analysis. Various design problems show that an extended tabulation is needed, and the present report provides this extension. The new tabulation (Appendix B) covers a non-dimensional frequency range of 0.00 to 0.70 in increments of 0.02 and a range of cavity fill ratios of 20 to 100 percent in maximum increments of ten percent.

A brief description of Stewartson's analysis is also given in this report. Emphasis is placed on the physical significance of the assumptions and results of the theory rather than its mathematical detail. The intent is to provide the novice designer of liquid-filled shell with an appreciation and first working knowledge of the analysis. Additionally, the summary of the theory is used to point out the significant advances that have been made in understanding liquid-filled shell problems since the publication of Ballistic Research Laboratories Memorandum Report No. 1477.

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	ROLE	WT	ROLE	WT	ROLE	WT
Liquid-Filled Projectiles Projectile Flight Stability Flight Mechanics						

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